数理解析研究所講究録
第2016巻 2017年 148-153

On the SL$(m, \mathbf{C})$-representation algebras of free groups and the Johnson homomorphisms

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Abstract

In this article, we consider certain descending filtrations of the SL$(m, \mathbf{C})$-representation algebras of free groups and free abelian groups. By using it, we introduce analogs of the Johnson homomorphisms of the automorphism groups of free groups. We show that the first homomorphisms are extended to the automorphism groups of free groups as crossed homomorphisms. Furthermore we show that the extended crossed homomorphisms induce Kawazumi's cocycles and Morita's cocycles. This works are generalization of our previous results [31] and [32] for the SL$(2, \mathbf{C})$-representation algebras.

For any $m \geq 2$ and any group G, let $R^m(G)$ be the set $\mathrm{Hom}(G, \mathrm{SL}(m, \mathbf{C}))$ of all SL$(m, \mathbf{C})$-representations of G. Let $\mathcal{F}(R^m(G), \mathbf{C})$ be the set $\{\chi : R^m(G) \to \mathbf{C}\}$ of all complex-valued functions on $R^m(G)$. Then $\mathcal{F}(R^m(G), \mathbf{C})$ naturally has the $\mathbf{C}$-algebra structure coming from the pointwise sum and product. For any $x \in G$ and any $1 \leq i, j \leq m$, we define an element $a_{ij}(x)$ of $\mathcal{F}(R^m(G), \mathbf{C})$ to be

$$(a_{ij}(x))(\rho) := (i, j)\text{-component of } \rho(x)$$

for any $\rho \in R^m(G)$. We call the map $a_{ij}(x)$ the (i, j)-component function of x, or simply a component function of x. Let $\mathfrak{R}^m_{\mathbf{Q}}(G)$ be the $\mathbf{Q}$-subalgebra of $\mathcal{F}(R^m(G), \mathbf{C})$ generated by all $a_{ij}(x)$ for $x \in G$ and $1 \leq i, j \leq m$. We call $\mathfrak{R}^m_{\mathbf{Q}}(G)$ the SL$(m, \mathbf{C})$-representation rings of G over $\mathbf{Q}$. In this article, we introduce a descending filtration of $\mathfrak{R}^m_{\mathbf{Q}}(G)$ consisting of $\mathrm{Aut}\, G$-invariant ideals, and study the graded quotients of it.

To the best of our knowledge, the study of the algebra $\mathfrak{R}^m_{\mathbf{Q}}(G)$ has a not so long history. Classically, the $\mathbf{Q}$-subalgebra of $\mathfrak{R}^2_{\mathbf{Q}}(F_n)$ generated by characters of F_n was actively studied. For any $x \in F_n$, the map $\mathrm{tr}\, x := a_{11}(x) + a_{22}(x)$ is called the Fricke character of x. Fricke and Klein [6] used the Fricke characters for the study of the classification of Riemann surfaces. In the 1970s, Horowitz [11] and [12] investigated several algebraic properties of the ring of Fricke characters by using the combinatorial

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group theory. In 1980, Magnus [19] studied some relations among Fricke characters of free groups systematically. As is found in Acuna-Maria-Montesinos's paper [1], today Magnus's research has been developed to the study of the SL(2, $\mathbf{C}$)-character varieties of free groups by quite many authors. Let $\mathfrak{X}^2_{\mathbf{Q}}(F_n)$ be the $\mathbf{Q}$-subalgebra of $\mathfrak{R}^2_{\mathbf{Q}}(F_n)$ generated by all $\mathrm{tr}\, x$ for $x \in F_n$. The ring $\mathfrak{X}^2_{\mathbf{Q}}(F_n)$ is called the ring of Fricke characters of F_n. Let $\mathfrak{C}$ be the ideal of $\mathfrak{X}^2_{\mathbf{Q}}(F_n)$ generated by $\mathrm{tr}\, x - 2$ for any $x \in F_n$. In our previous papers [9] and [30], we considered an application of the theory of the Johnson homomorphisms of $\mathrm{Aut}\, F_n$ by using the Fricke characters. In particular, we determined the structure of the graded quotients $\mathrm{gr}^k(\mathfrak{C}) := \mathfrak{C}^k/\mathfrak{C}^{k+1}$ for $1 \leq k \leq 2$, introduced analogs of the Johnson homomorphisms, and showed that the first homomorphism extends to $\mathrm{Aut}\, F_n$ as a crossed homomorphism.

We briefly review the history of the Johnson homomorphisms. In 1965, Andreadakis [2] introduced a certain descending central filtration of $\mathrm{Aut}\, F_n$ by using the natural action of $\mathrm{Aut}\, F_n$ on the nilpotent quotients of F_n. We call this filtration the Andreadakis-Johnson filtration of $\mathrm{Aut}\, F_n$. In the 1980s, Johnson studied such filtration for mapping class groups of surfaces in order to investigate the group structure of the Torelli groups in a series of works [13], [14], [15] and [16]. In particular, he determined the abelianization of the Torelli group by introducing a certain homomorphism. Today, his homomorphism is called the first Johnson homomorphism, and it is generalized to higher degrees. Over the last two decades, the Johnson homomorphisms of the mapping class groups have been actively studied from various viewpoints by many authors including Morita [21], Hain [8] and others.

The Johnson homomorphisms are naturally defined for $\mathrm{Aut}\, F_n$:

$$\tilde{\tau}_k : \mathcal{A}_{F_n}(k) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_{F_n}(k+1))$$

where $H := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. So far, we concentrate on the study of the cokernels of Johnson homomorphisms of $\mathrm{Aut}\, F_n$ in a series of our works [25], [26], [28] and [5] with combinatorial group theory and representation theory. Since each of the Johnson homomorphisms is $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective, it is an important problem to determine its images and cokernels. On the other hand, the study of the extendability of the Johnson homomorphisms have been received attentions. Morita [22] showed that the first Johnson homomorphism of the mapping class group, which initial domain is the Torelli group, can be extended to the mapping class group as a crossed homomorphism by using the extension theory of groups. Inspired by Morita's work, Kawazumi [17] obtained a corresponding results for $\mathrm{Aut}\, F_n$ by using the Magnus expansion of F_n. Furthermore, he constructed higher twisted cohomology classes with the extended first Johnson homomorphism and the cup product. By restricting them to the mapping class group, he investigated relations between the higher cocycles and the Morita-Mumford classes. Recently, Day [4] showed that each of Johnson homomorphisms of $\mathrm{Aut}\, F_n$ can be extended to a crossed homomorphism from $\mathrm{Aut}\, F_n$ into a certain finitely generated free abelian group.

As mentioned above, we [9] constructed analogs of the Johnson homomorphisms with the ring $\mathfrak{X}^2_{\mathbf{Q}}(F_n)$ of Fricke characters, and showed that the first homomorphism can be extended to $\mathrm{Aut}\,F_n$ in [30]. It is, however, difficult to push forward with our research since the structures of the graded quotients $\mathrm{gr}^k(\mathfrak{C})$ are too complicated to handle. In [31], we considered a similar situation for the $\mathrm{SL}(2,\mathbf{C})$-representation algebra $\mathfrak{R}^2_{\mathbf{Q}}(F_n)$. In this article, we generalize our previous works to the $\mathrm{SL}(m,\mathbf{C})$-representation case. Set $s_{ij}(x) := a_{ij}(x) - \delta_{ij}$ for any $1 \leq i,j \leq m$ and $x \in F_n$ where δ means Kronecker's delta. Let $\mathfrak{J}_{F_n}$ be the ideal of $\mathfrak{R}^m_{\mathbf{Q}}(F_n)$ generated by $s_{ij}(x_l)$ for any $1 \leq i,j \leq m$ and $1 \leq l \leq n$. Then the products of $\mathfrak{J}_{F_n}$ define a descending filtration of $\mathfrak{R}^m_{\mathbf{Q}}(F_n)$:

$$\mathfrak{J}_{F_n} \supset \mathfrak{J}^2_{F_n} \supset \mathfrak{J}^3_{F_n} \supset \cdots,$$

which consists of $\mathrm{Aut}\,F_n$-invariant ideals. Set $\mathrm{gr}^k(\mathfrak{J}_{F_n}) := \mathfrak{J}^k_{F_n}/\mathfrak{J}^{k+1}_{F_n}$ for any $k \geq 1$. Set

$$T_k := \Big\{ \prod_{\substack{1\leq i,j\leq m \\ (i,j)\neq(m,m)}} \prod_{l=1}^{n} s_{ij}(x_l)^{e_{ij,l}} \,\Big|\, e_{ij,l} \geq 0, \quad \sum_{\substack{1\leq i,j\leq m \\ (i,j)\neq(m,m)}} \sum_{l=1}^{n} e_{ij,l} = k \Big\} \subset \mathfrak{J}^k_{F_n}.$$

Theorem 1. *For each $k \geq 1$, the set $T_k \pmod{\mathfrak{J}^{k+1}_{F_n}}$ forms a basis of $\mathrm{gr}^k(\mathfrak{J}_{F_n})$ as a $\mathbf{Q}$-vector space. Furthermore, for any $n \geq 2$ and $k \geq 1$, we have*

$$\mathrm{gr}^k(\mathfrak{J}_{F_n}) \cong {\bigoplus}' \bigotimes_{\substack{1\leq i,j\leq m \\ (i,j)\neq(m,m)}} S^{e_{ij}} H_{\mathbf{Q}}$$

as a $\mathrm{GL}(n,\mathbf{Z})$-module. Here the sum runs over all tuples (e_{ij}) for $1 \leq i,j \leq m$ and $(i,j) \neq (m,m)$ such that the sum of the e_{ij} is equal to k.

This theorem is a generalization of our previous result for the case where $k = 2$ in [31].

Now, set

$$\mathcal{D}^m_{F_n}(k) := \mathrm{Ker}(\mathrm{Aut}\,F_n \to \mathrm{Aut}(\mathfrak{J}_{F_n}/\mathfrak{J}^{k+1}_{F_n})).$$

The groups $\mathcal{D}^m_{F_n}(k)$ define a descending central filtration of $\mathrm{Aut}\,F_n$. Let $\mathcal{A}_{F_n}(1) \supset \mathcal{A}_{F_n}(2) \supset \cdots$ be the Andreadakis-Johnson filtration of $\mathrm{Aut}\,F_n$. We show a relation between $\mathcal{A}_{F_n}(k)$ and $\mathcal{D}^m_{F_n}(k)$, and among $\mathcal{D}^m_{F_n}(k)$s as follows.

Theorem 2.

(1) *For any $k \geq 1$, $\mathcal{A}_{F_n}(k) \subset \mathcal{D}^m_{F_n}(k)$.*

(2) *For any $k \geq 1$ and $m \geq 2$, we have $\mathcal{D}^{m+1}_{F_n}(k) \subset \mathcal{D}^m_{F_n}(k)$.*

In [31], we showed that $\mathcal{D}^2_{F_n}(k) = \mathcal{A}_{F_n}(k)$ for $1 \leq k \leq 4$. Hence, we have $\mathcal{D}^m_{F_n}(k) = \mathcal{A}_{F_n}(k)$ for any $1 \leq k \leq 4$. By referring to the theory of the Johnson homomorphisms, we can construct analogs of them:

$$\tilde{\eta}_k : \mathcal{D}^m_{F_n}(k) \to \mathrm{Hom}_{\mathbf{Q}}(\mathrm{gr}^1(\mathfrak{J}_{F_n}), \mathrm{gr}^{k+1}(\mathfrak{J}_{F_n})).$$

defined by the corresponding $f \mapsto f^\sigma - f$ for any $f \in \mathfrak{J}_{F_n}$. In this article, we consider an extension of the first homomorphism $\tilde{\eta}_1$, and study some relations to the extension of $\tilde{\tau}_1$.

Set $H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q}$. In [24], we computed $H^1(\mathrm{Aut}\, F_n, H_{\mathbf{Q}}) = \mathbf{Q}$, and showed that it is generated by Morita's cocycle f_M. On the other hand, we [27] also computed $H^1(\mathrm{Aut}\, F_n, H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}}) = \mathbf{Q}^{\oplus 2}$, and showed that it is generated by Kawazumi's cocycle f_K and the cocycle induced from f_M. Kawazumi's cocycle f_K is an extension of $\tilde{\tau}_1$. Then we can construct the crossed homomorphism

$$\theta_{F_n} : \mathrm{Aut}\, F_n \to \mathrm{Hom}_{\mathbf{Q}}(\mathrm{gr}^1(\mathfrak{J}_{F_n}), \mathrm{gr}^2(\mathfrak{J}_{F_n}))$$

which is an extension of $\tilde{\eta}_1$. By taking suitable reductions of the target of θ_{F_n}, we obtain the crossed homomorphisms

$$f_1 : \mathrm{Aut}\, F_n \to H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}}, \quad f_2 : \mathrm{Aut}\, F_n \to H_{\mathbf{Q}}.$$

Then we show the following.

Theorem 3. *For any $n \geq 2$,*

$$f_K = f_1, \quad f_M = -f_2 + \delta_x$$

for $x = x_1 + x_2 + \cdots + x_n \in H_{\mathbf{Q}}$ as crossed homomorphisms.

This shows that our crossed homomorphism induces both of Kawazumi's cocycle and Morita's cocycle, and that θ_{F_n} defines the non-trivial cohomology class in $H^1(\mathrm{Aut}\, F_n, \mathrm{Hom}_{\mathbf{Q}}(\mathrm{gr}^1(\mathfrak{J}_{F_n}), \mathrm{gr}^2(\mathfrak{J}_{F_n})))$.

In [10] and [32], we studied $\mathfrak{X}_{\mathbf{Q}}^2(H)$ and $\mathfrak{R}_{\mathbf{Q}}^2(H)$. In this paper, we generalize the results in [32] to the $\mathrm{SL}(m, \mathbf{C})$-representation cases. By using a parallel argument, we can define a descending filtration $\mathfrak{J}_H \supset \mathfrak{J}_H^2 \supset \mathfrak{J}_H^3 \supset \cdots$ of ideals in $\mathfrak{R}_{\mathbf{Q}}^2(H)$. In contrast with the free group case, however, it is a quite hard to determine the structure of the graded quotients $\mathrm{gr}^k(\mathfrak{J}_H)$. Here, we gave basis of $\mathrm{gr}^k(\mathfrak{J}_H)$ for $1 \leq k \leq 2$. In particular, we see

$$\mathrm{gr}^1(\mathfrak{J}_H) \cong H_{\mathbf{Q}}^{\oplus m^2-1}, \quad \mathrm{gr}^2(\mathfrak{J}_H) \cong (S^2 H_{\mathbf{Q}})^{\oplus \frac{1}{2}m^2(m^2-1)} \oplus (\Lambda^2 H_{\mathbf{Q}})^{\{\oplus \frac{1}{2}(m^2-1)(m^2-4)\}}.$$

We remark that for a general $m \geq 3$, the situation of the $\mathrm{SL}(m, \mathbf{C})$-representation case is much more different and complicated than those of the $\mathrm{SL}(2, \mathbf{C})$-representation case. At the present stage, we have no idea to give a result for a general $k \geq 3$. By using the above results, we construct the crossed homomorphism

$$\theta_H : \mathrm{Aut}\, F_n \to \mathrm{Hom}_{\mathbf{Q}}(\mathrm{gr}^1(\mathfrak{J}_H), \mathrm{gr}^2(\mathfrak{J}_H)),$$

and show that it induces Morita's cocycle f_M.

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